These combined with (1) give

$$v_x' = u \sin\theta \, \frac{a^2 - \alpha k^2}{a^2 + k^2}, \qquad v_y' = + u\alpha \cos\theta. \tag{2}$$

The equations of motion are $\ddot{x} = g \sin\theta$, $\ddot{y} = - g\cos\theta$, or $\dot{x} = gt \sin\theta + c_1$, $\dot{y} = - gt\cos\theta + c_2$. When $t = 0$, $\dot{x} = v_x'$ and $\dot{y} = v_y'$. Hence determining c_1 and c_2 from (2), we get

$$\dot{x} = \sin\theta\left(gt + u\frac{a^2 - \alpha k^2}{a^2 + k^2}\right), \qquad \dot{y} = \cos\theta\,(u\alpha - gt).$$

Integrating again, with the conditions $x = 0$ and $y = a$ when $t = 0$,

$$x = \sin\theta\left(\tfrac{1}{2}gt^2 + tu\frac{a^2 - \alpha k^2}{a^2 + k^2}\right), \qquad y = a + \cos\theta\,(u\alpha t - \tfrac{1}{2}gt^2).$$

The ball hits the plane a second time when $y = a$, i.e. when $t = 2u\alpha/g$. The corresponding value of x becomes, after simplification,

$$\frac{2a^2\alpha(1 + \alpha)u^2}{g(a^2 + k^2)} \sin\theta.$$

Also solved by H. L. Olson, Arthur Pelletier, and J. B. Reynolds.

2860 [1920, 428]. Proposed by E. O. BROWN, Chicago, Ill.

A frustum of a right circular cone has a volume v. The lateral area added to the lesser base is a sum which is a minimum. Determine the dimensions of the frustum in terms of v.

Solution by H. S. Uhler, Yale University.

Let h, r, R, and s denote respectively the altitude, the radius of the smaller base, the radius of the larger base, and the slant height of the frustum. Also let σ symbolize the sum in question. Then

$$\sigma = \pi(Rs + rs + r^2), \tag{1}$$

$$v = \tfrac{1}{3}\pi h(R^2 + Rr + r^2), \tag{2}$$

$$s = + [h^2 + (R - r)^2]^{1/2}. \tag{3}$$

By virtue of equations (2) and (3), σ may be considered as a function of the two independent variables R and r, hence the necessary conditions for a minimum of σ are

$$\frac{\partial\sigma}{\partial R} = 0, \qquad \frac{\partial\sigma}{\partial r} = 0.$$

Differentiating equations (1), (2), and (3) with respect to R and combining the results we obtain

$$\frac{\partial\sigma}{\partial R} = \frac{\pi}{s}\left\{ s^2 + (R + r)\left[R - r - \frac{9v^2(2R + r)}{\pi^2(R^2 + Rr + r^2)^3}\right]\right\} = 0,$$

whence

$$\frac{9v^2}{\pi^2(R^2 + Rr + r^2)^3} = \frac{2(R - r)}{R + 2r}. \tag{4}$$

In like manner we find

$$\frac{\partial\sigma}{\partial r} = \frac{\pi r}{s}\left[2s - 2(R - r) - \frac{9v^2(2R + r)}{\pi^2(R^2 + Rr + r^2)^3}\right] = 0. \tag{5}$$

The ratio $r/R \equiv \rho$ may be obtained in the following manner. First substitute the equal of the left member of equation (4) in the third or fractional term of equation (5) to get

$$s = \frac{3(R^2 - r^2)}{R + 2r}. \tag{6}$$

Then square the last equation, eliminate v^2 by again using relation (4), and suppress the com-

mon factor $R - r$ to obtain $3\rho^3 + 5\rho^2 - 2 = 0$, the roots of this equation being

$$-1, \qquad -(\sqrt{7}+1)/3, \qquad (\sqrt{7}-1)/3.$$

Substituting the positive root in the following modified form of equation (4)

$$R^6 = \frac{9(1+2\rho)v^2}{2(1-\rho^3)(1+\rho+\rho^2)^2\pi^2},$$

we get

$$R = \frac{(88+13\sqrt{7})^{1/6}v^{1/3}}{2^{1/6}\cdot 7^{1/4}\pi^{1/3}} \doteq 0.833293v^{1/3},$$

and $r = \frac{1}{3}(\sqrt{7}-1)R \doteq 0.457131v^{1/3}$.

The values of h and s may now be obtained from formulas (2) and (6) respectively. They are $h = [2(\sqrt{7}-2)v/\pi]^{1/3} \doteq 0.743559v^{1/3}$, $s = R$.

The last relation is the most significant geometrically. Let θ denote the acute angle between the axis and the slant height of the frustum. Then

$$\theta = \sin^{-1}[(R-r)/s] = \sin^{-1}[(4-\sqrt{7})/3] \doteq 26° \; 50' \; 4.5''.$$

From the practical point of view, the problem amounts to asking for the smallest amount of sheet metal out of which a cup or deep cake pan of given capacity can be made.

2870 [1921, 36]. Proposed by WARREN WEAVER, University of Wisconsin.

A pendulum bob of mass m is attached to one end of a weightless and inextensible string of length l and swings as a conical pendulum with an angular velocity ω_1 about a vertical line through a fixed point to which the other end of the string is attached. If the angular velocity is increased to ω_2, the height through which the bob rises is independent of the length l. Consider, then, a very long and a very short pendulum. Suppose they are each swung first with an angular velocity ω_1 and then with a larger angular velocity, ω_2, the difference between these two values being great enough so that the longer pendulum rises through a height greater than the length of the shorter pendulum. According to the above result, the shorter one should rise through this same height, which is obviously impossible. Explain this apparent paradox.

SOLUTION BY I. MAIZLISH, University of Minnesota.

Let l be the length of the string, α_1 the angle which the equilibrium position of the pendulum makes with the vertical when the angular velocity is ω_1, and α_2 the corresponding angle when the angular velocity is ω_2. Assume $\omega_2 > \omega_1$, that the pendulum is free to take a vertical position when not revolving, and that ω_1 and ω_2 are finite. It is readily seen that for equilibrium the following equation must hold:

$$\tan\alpha_1 = (\omega_1^2 l \sin\alpha_1)/g, \tag{1}$$

and if $\alpha_1 > 0$ we must have

$$\omega_1 > \sqrt{(g/l)}. \tag{2}$$

The height of the bob from the horizontal plane passing through it when the pendulum is in its vertical position is $h_1 = l(1-\cos\alpha_1) = l - (g/\omega_1^2)$, and if the angular velocity is increased from ω_1 to ω_2, α being increased to α_2, the height through which the bob rises will be

$$h_2 - h_1 = (g/\omega_1^2) - (g/\omega_2^2), \tag{3}$$

as long as $\omega_1 \geqq \sqrt{(g/l)}$.

Let l' be the length of the short pendulum. If ω_1 and ω_2 are chosen so that $(g/\omega_1^2) - (g/\omega_2^2) > l'$ we shall have $\omega_1 < \sqrt{(g/l')}$ and we cannot apply (3) to this pendulum. If $\omega_2 > \sqrt{(g/l')}$ and the bob of the short pendulum rises at all, the height to which it rises will be $l' - g/\omega_2^2$, but if $\omega_2 \leqq \sqrt{(g/l')}$, it will not rise at all.

Also solved by H. L. OLSON, F. L. WILMER and the PROPOSER.

2872 [1921, 36]. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

The rectangular coördinates of a point P at the time t are given by the equations

$$x = k\cos\gamma\cos(nt-\alpha), \qquad y = k\sin\gamma\cos(nt-\beta),$$